Deriving the quadratic formula

Consider the quadratic equation in the form $a_0 + a_1 z + a_2 z^2 = 0$, where $a_2 \neq 0$.

1. Multiply both sides of the equation by $4a_2$:

$$4a_0a_2 + 4a_1a_2z + 4a_2^2z^2 = 0$$

2. Substitute $z \leftarrow x - \frac{a_1}{2a_2}$ and expand:

$$\begin{aligned} 4a_0a_2 + 4a_1a_2\left(x - \frac{a_1}{2a_2}\right) + 4a_2^2\left(x - \frac{a_1}{2a_2}\right)^2 &= 0 \\ 4a_0a_2 + \left(4a_1a_2x - 2a_1^2\right) + \left(4a_2^2x^2 - 4a_1a_2x + a_1^2\right) &= 0 \\ 4a_0a_2 - a_1^2 + 4a_2^2x^2 &= 0 \end{aligned}$$

3. Add $a_1^2 - 4a_0a_2$ to both sides to isolate the squared term:

$$4a_2^2x^2 = a_1^2 - 4a_0a_2$$

4. Take the square root of both sides:

$$2a_2x = \sqrt{a_1^2 - 4a_0a_2}$$

5. Solve for x by dividing both sides by $2a_2$:

$$x = \frac{\sqrt{a_1^2 - 4a_0a_2}}{2a_2}$$

Since every non-zero complex number has two distinct square roots (which are negatives of each other), back-substituting x yields the two solutions to the quadratic equation:

$$z = \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_2} \text{ and } z = \frac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_2}.$$